

A LOCAL FIELD APPROACH TO THE RIEMANN HYPOTHESIS

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ABSTRACT. Since the seminal work of Wan, Poonen, and Sheats in the 1990's, we have been searching for the correct general statement of the Riemann Hypothesis ("RH") which appears implicit in their results. Recently, upon viewing the extension \mathbb{C}/\mathbb{R} in light of results derived for the Carlitz module, we were led to view the RH as a statement about ramification which we explore in this short work. We shall see that, combined with some ideas flowing from the proofs of Wan, Poonen, and Sheats, this ramification idea has a good deal of explanatory power in finite characteristic. Indeed, unramified extensions of nonarchimedean local fields are cyclotomic in nature and this fits perfectly with the best possible extension of the result of Wan and Sheats. The notion that the zeroes "lie on a line" seems to be the beginning of the story in finite characteristic, and we show how this fits with having the zeroes be unramified.

1. INTRODUCTION

In 1996, in response to a query from the author, Daqing Wan [Wa1] calculated the Newton polygon associated to the characteristic zeta function associated to $\mathbb{F}_p[\theta]$. What he discovered was completely remarkable: there is at most one (with multiplicity) zero of a given absolute value. As an obvious and immediate corollary, one finds that the zeroes of the zeta function (at the infinite prime) are both simple and lie "on the line" $\mathbb{F}_p((1/\theta))$. Clearly this is both a "Riemann Hypothesis," as the zeroes lie on a "line", and the expected analog of the simplicity of zeroes of the Riemann zeta function.

Since that time, due to the labors of D. Thakur, B. Poonen, J. Diaz-Vargas and J. Sheats [DV1, Sh1, Ba1], Wan's result has been extended to the general polynomial ring $\mathbb{F}_q[\theta]$ with the same results. While we do not yet have the theory to understand the arithmetic applications of this result, one clearly wants to put it into an appropriate framework for general L -series. We tried this originally in [Go2] which is now superseded by the present work (where we are more specific about the fields generated by the zeroes; moreover in [Go2] we were not careful enough about the factorizations of zeta functions in terms of L -series as discussed below in Subsection 4.2).

The zeta-function of $\mathbb{F}_q[\theta]$ is known to have an analytic continuation as a family $\zeta(x, y)$ of entire functions in x^{-1} , where $y \in \mathbb{Z}_p$ and \mathbb{Z}_p is the p -adic integers, see e.g., Section 8 of [Go1], [Bo1] or [BP1]. In fact, one finds that *all* L -series of Drinfeld modules defined over finite extensions $L/\mathbb{F}_q(\theta)$ have similar analytic continuations. So a first guess at a "generalized Riemann Hypothesis," would be that all the zeroes of such L -series lie in $\mathbb{F}_q((1/\theta))$. On the other hand, nonArchimedean analysis is very algebraic in nature and every entire function is completely determined by its zeroes (unlike, obviously, the exponential function classically). We shall see in Section 4.2 that this immediately leads to counterexamples of this first guess.

Recently upon viewing the extension \mathbb{C}/\mathbb{R} from the point of view of the Carlitz module, as in Subsection 2 below, we have come to some clarity on these issues. Because of this

insight, we view the classical Riemann hypothesis as the statement that the zeta zeroes are “unramified”. We will show how, in finite characteristic, this fits very nicely into the framework initiated by Wan and this will lead to speculation on the nature of “most” zeta zeroes (and perhaps all in certain circumstances). Indeed, combined with the calculations of Wan/Poonen/Sheats, this new insight has some serious predictive power and actually leads to the best possible guess on the nature of the zeroes in view of the above counterexamples.

One of the basic statements about the Riemann zeta function is, of course, its famous functional equation. It has long been known that a functional equation of the same $s \mapsto 1-s$ type does not appear to hold with characteristic p zeta functions. However, in [Go4] we introduced the group $S_{(q)}$ of self-homeomorphisms of \mathbb{Z}_p (essentially $S_{(q)}$ is the permutation group on $\{0, 1, 2, \dots\}$ acting on \mathbb{Z}_p by simply permuting the q -adic coefficients). Remarkably, there is also *some* evidence that $S_{(q)}$ acts on the zeta zeroes themselves via their expansions as elements of $\mathbb{F}_q((1/\theta))$. As this action in finite characteristic is in fact linear, we will see in Subsection 5.1 that it also works well with the new optic of “unramified” zeroes.

We emphasize again that we are speculating here and actually establish very little. However, it is our goal to present a framework to view the original Riemann Hypothesis which *also* leads to important ideas for characteristic p valued L -series and which, perhaps, gives a suitable framework for future exploration.

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1.1. Notation. We set $q = p^{m_0}$ where p is prime and m_0 is a nonnegative integer, and let \mathbb{F}_q be the field with q elements. We set $A := \mathbb{F}_q[\theta]$, $k := \mathbb{F}_q(\theta)$, and $K := \mathbb{F}_q((1/\theta))$. Notice that K is the completion of k at the normalized absolute value associated to the infinite place of k . Let \bar{K} be a fixed algebraic closure of K and let $\bar{\mathbb{F}}_q \subset \bar{K}$ be the algebraic closure of \mathbb{F}_q .

2. \mathbb{C}/\mathbb{R} FROM THE VIEWPOINT OF THE CARLITZ MODULE

It has always been an interesting game whether the extension \mathbb{C}/\mathbb{R} should be viewed as (somehow) ramified or not; of course, as $[\mathbb{C}:\mathbb{R}] = 2$, *if* one views this extension as ramified, then the ramification should be viewed as total.

One piece of evidence that points to this extension being unramified is that the value group of the usual archimedean absolute value $|\cdot|$ is the same on both fields, and this is a hallmark of an unramified extension (of nonArchimedean local fields)...

On the other hand, the theory of function fields over a finite field contains a number of analogs of the extension \mathbb{C}/\mathbb{R} . Indeed, one analog would be our chosen algebraic closure \bar{K} of K ; this is clearly algebraically closed, but is not complete. So another analog of \mathbb{C} is the field \mathbb{C}_∞ given by the completion of \bar{K} via the canonical extension of the absolute value $|\cdot|_\infty$. The field \mathbb{C}_∞ is then both complete *and* algebraically closed. However, it is *not* a local field (i.e., it is not locally compact); so the analogy is not exact.

However, the Carlitz module ([Ca1], or Section 3 of [Go1]) immediately presents yet another analog of \mathbb{C} that we call K_1 and which, in our view, mandates viewing \mathbb{C}/\mathbb{R} as ramified. Let \exp be the (classical) exponential function and let \exp_C be the exponential function of the Carlitz module. Recall that $2\pi i$ is, of course, the period of \exp and $\mathbb{C} = \mathbb{R}(2\pi i)$. Early on, Carlitz gave a beautiful formula for the “period” $\tilde{\xi}$ of \exp_C which was remarkably simplified in [AT1] in the following fashion. Let θ_1 be a fixed choice of $(q-1)$ -st root of $-\theta$.

Then one has

$$\tilde{\xi} = \theta_1 \theta \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}. \quad (1)$$

Definition 1. We set $K_1 := K(\tilde{\xi})$.

Obviously K_1 is a local field and, from the present viewpoint, is thus clearly an analog of \mathbb{C} (though by no means is it algebraically closed). Also clearly, K_1/K is a *totally (and tamely) ramified extension* with Galois group isomorphic to $\mathbb{F}_q^* = A^*$ (much as \mathbb{C}/\mathbb{R} has Galois group isomorphic to \mathbb{Z}^*).

Convention 1. From now on, we shall view \mathbb{C}/\mathbb{R} as a totally ramified extension.

Remark 1. Let L be a nonArchimedean local field with residue characteristic p . It is then universally known that the unramified extensions of L are given by adjoining n -th power roots of unity with n not divisible by p .

An element α algebraic over a local field L will be called “unramified” if and only if the extension $L(\alpha)/L$ is unramified.

3. THE CLASSICAL RIEMANN HYPOTHESIS AS A RAMIFICATION STATEMENT

Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function. Riemann showed that $\zeta(s)$ has a meromorphic continuation to \mathbb{C} . Let $\Gamma(s)$ be Euler’s gamma function and put

$$\Xi(s) := 1/2\pi^{-s/2} s(s-1)\Gamma(s/2)\zeta(s). \quad (2)$$

Riemann then showed that $\Xi(s)$ is entire and satisfies the functional equation $\Xi(s) = \Xi(1-s)$. The *Riemann Hypothesis* is precisely the statement that the nontrivial zeroes of $\Xi(s)$ lie on the line $1/2 + it$ where $t \in \mathbb{R}$. Following Riemann, we put $\tilde{\Xi}(t) := \Xi(1/2 + it)$ so that the Riemann Hypothesis is now the statement that the zeroes of $\tilde{\Xi}(t)$ are *real*.

Convention 2. In keeping with Convention 1, we rephrase the Riemann Hypothesis as the statement that the zeroes of $\tilde{\Xi}(t)$ are *unramified*.

4. THE RH AS A RAMIFICATION STATEMENT IN CHARACTERISTIC p

4.1. The results of Wan and Sheats. Let A etc., be as in Subsection 1.1. Let A' be the set of monic elements of A and formally put

$$\zeta_A(s) = \zeta(s) = \sum_{a \in A'} a^{-s}. \quad (3)$$

It is clear, for instance, that this series will converge to an element of K if s is a positive integer. It has been shown in various places (see, for instance, Section 8 of [Go1]) that $\zeta(s)$ has an analytic continuation to an “entire” function $\zeta(x, y)$ on the space $\mathbb{S}_{\infty} := \mathbb{C}_{\infty}^* \times \mathbb{Z}_p$, where \mathbb{C}_{∞}^* was defined in Section 2 above. In practical terms this means that for each $y \in \mathbb{Z}_p$ we obtain an entire power series $\zeta(x, y)$ in x^{-1} with coefficients in K .

Theorem 1. Fix $y \in \mathbb{Z}_p$. Then each zero of $\zeta(x, y)$ is uniquely determined by its absolute value (including multiplicity).

Theorem 1 is the result of Wan ($q = p$) [Wa1], Poonen, and Sheats (arbitrary q) [Sh1], and the next result then follows immediately.

Corollary 1. *The zeroes of $\zeta(s)$ lie in K and are simple.*

We view Theorem 1 as the “Riemann Hypothesis” for $\zeta(s)$. Obviously as elements of K , the zeroes of $\zeta(s)$ are trivially unramified over K .

4.2. The failure of the naive generalization of Wan/Sheats. Let L/k be a finite abelian extension of order β prime to p and exponent greater than p ; such extensions are very easy to construct. Let $G := \text{Gal}(L/k)$ and let \mathfrak{O} be the ring of A -integers of L .

Using the monic generators of the ideal norms from \mathfrak{O} to A , one is easily able to define the analog $\zeta_{\mathfrak{O}}(s)$ of the *Dedekind zeta function* of \mathfrak{O} . It is easy to see that $\zeta_{\mathfrak{O}}(x, y)$ also analytically continues to a family of entire functions with coefficients in the local field K .

Let \bar{K} be an algebraic closure of K as in Section 2 and let $\psi: G \rightarrow \bar{K}^*$ be a character. As $[L: k]$ is prime to p , it does not matter whether we view ψ as having values in \bar{K} or in \mathbb{C} . Using the formalism of the classical zeta factorizations (i.e., we use the same local polynomials but then reduce them modulo p), one obtains the expected factorization

$$\zeta_{\mathfrak{O}}(s) = \prod_{\psi} L(\psi, s), \quad (4)$$

where $L(\psi, s)$ has the obvious definition. The functions $L(\psi; x, y)$ also have analytic continuations as entire power series in analogy with $\zeta(s)$.

N.B.: Let ζ_{β} be a primitive β -th root of unity and let $K_{\beta} = K(\zeta_{\beta})$ be the unramified extension of K obtained by adjoining ζ_{β} . One sees that $L(\psi, s)$ gives rise to a family of entire power series in x^{-1} with coefficients in K_{β} . In fact, one can readily find examples where these coefficients do *not* lie in K itself.

As $[K_{\beta}: K] < \infty$, the field K_{β} is automatically complete. On the other hand, for fixed $y \in \mathbb{Z}_p$, the entire function $L(\psi; x, y)$ factors over its zeroes. If all such zeroes were contained in K , then the coefficients must also be in K and we have seen this is impossible in general.

4.3. The “unramified Riemann hypothesis” in finite characteristic. Continuing with the above discussion, the best that could be hoped for in terms of the zeroes of $L(\psi; x, y)$ is that they lie in the field K_{β} and so in particular are unramified over K . We state this as a question.

Question 1. Are the zeroes of $L(\psi, s)$ (and thus also $\zeta_{\mathfrak{O}}(s)$) unramified over K ?

We view the lack of ramification of the zeroes as the appropriate analog of the Riemann hypothesis for such finite characteristic valued functions.

Remark 2. A completely similar analysis works when L/k is an arbitrary, not necessarily abelian, Galois extension of finite order β prime to p . We will discuss the case where the order is divisible by p in Subsection 5.2 below.

Merely stipulating that the zeroes are unramified does not guarantee that they lie in a finite extension. However, the next question is very natural in light of Wan/Sheats.

Question 2. Let L/k be finite Galois of order β prime to p . Let ρ be an irreducible representation of $\text{Gal}(L/k)$ and let $L(\rho; x, y)$ be the associated L -series (which we also know to be entire, etc.). Fix $y \in \mathbb{Z}_p$. Is every zero of $L(\rho; x, y)$ uniquely determined by its absolute value (including multiplicity)?

Note of course that if the answer to Question 2 is yes, then the zeroes of the L -series will be unramified, simple and lie in $K(\zeta_\beta)$, where $\beta = [L:k]$, and so are then obviously contained in a finite extension of K .

Remark 3. The simplicity and lack of ramification of such zeroes is also expected in the analogous classical case.

Remark 4. The factorizations into L -series, combined with positive answers to the above questions, would allow one to predict the fields obtained by adjoining the zeroes of the analogs of Dedekind zeta functions, etc.

5. COMPLEMENTS AND CAVEATS

5.1. The connection with the group $S_{(q)}$. Let q continue as above and let $y \in \mathbb{Z}_p$. Write y q -adically as

$$y = \sum_{i=0}^{\infty} c_i q^i \quad (5)$$

where $0 \leq c_i < q$ for all i .

Now let ρ be a permutation of the set $\{0, 1, 2, \dots\}$.

Definition 2. We define $\rho_*(y)$, $y \in \mathbb{Z}_p$, by

$$\rho_*(y) := \sum_{i=0}^{\infty} c_i q^{\rho(i)}, \quad (6)$$

where y is written as in Equation 5.

Clearly $y \mapsto \rho_*(y)$ gives a representation of ρ as a bijection of \mathbb{Z}_p which is denoted ρ_* . The group of such bijections is denoted $S_{(q)}$. In [Go4], it is shown that ρ_* is, in fact, a homeomorphism of \mathbb{Z}_p which stabilizes both the nonnegative and nonpositive integers lying inside \mathbb{Z}_p . Because of this stability, one is able to also define a corresponding action of ρ_* on K , as certain \mathbb{F}_q -linear continuous operators. In fact, ρ_* appears to also act on the zeta zeroes (though it must be admitted that there is so far only a bit of evidence for this).

It is by no means clear how, or even if, one can extend ρ_* to the algebraic closure \bar{K} . However, extending it linearly to $\bar{\mathbb{F}}_q$ gives the obvious natural extension to the unramified closure K^{un} of K , and thus fits well with the ideas of Subsection 4.3.

5.2. Caveats. In this final subsection we present some caveats related to Questions 1 and 2. First of all let L/k now be a finite abelian extension of order p with group G . Let $\mathfrak{O} \subset L$ be the A -integers as before. We assume, for simplicity, that L/k is ramified at one finite prime $\mathfrak{p} = (f)$, f monic – such examples are easy to find. Let $\zeta_{\mathfrak{O}}(s)$ be the associated analog of the Dedekind zeta function. Using the characteristic 0 factors as before we still have the factorization

$$\zeta_{\mathfrak{O}}(s) = \prod_{\psi} L(\psi, s).$$

where ψ runs through the characteristic 0 valued characters of G and where we then reduce the local factors modulo p . As the order of G is p , we then see immediately that if ψ is nontrivial, $L(\psi, s) = \zeta_A(s)(1 - f^{-s})$ giving the factorization

$$\zeta_{\mathfrak{O}}(s) = \zeta_A(s)^p (1 - f^{-s})^{p-1}.$$

Obviously the zeroes of the Euler factor $(1 - f^{-s})$ may even be inseparable depending on the degree of f . In these sort of cases, though, it may be possible to systematically remove these spurious zeroes through a closer examination of ramification.

On the other hand, we should expect all of this to work for Drinfeld's general rings A (= the affine algebra of a smooth projective geometrically connected curve over \mathbb{F}_q minus a chosen place ∞). However, in this generality, G. Böckle has examples where there are such spurious zeroes which are *not* attributable to issues involving Euler factors. Different techniques will thus be needed here to handle these.

Moreover, one would expect a similar theory to hold at the interpolations of these functions at finite primes f . Here one has to consistently remove Euler factors of the form $(1 - f^i x^{-d})$ where d is the degree of f in order to even do the interpolations. Again removing such Euler factors will contribute many spurious zeroes.

Thus, until a theory arises to handle the spurious zeroes, the general versions of Questions 1 and 2 – applying to all rings A and all places of the associated complete curve – should be modified to pertain to *almost all* zeroes for a given $y \in \mathbb{Z}_p$, etc.

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